Math 10A
Worksheet, Midterm II Review; Thursday, 7/19/2018
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1 FTC

1. **TRUE** False $\int_0^x e^{t^2} dt$ is an antiderivative of e^{x^2}.

Solution: This is true by FTC II. The derivative of $\int_a^x e^{t^2} dt$ is e^{x^2} by FTC II. But, there is no way to write it in terms of elementary functions.

2. If $\int_0^x f(t)dt = \frac{1}{2}\cos(2x) - a$, find f, a.

Solution: Take the derivative of both sides with respect to x to get that $f(x) = -\sin(2x)$. Then we plug this back in to get

$$\int_0^x f(t)dt = \int_0^x -\sin(2t)dt = \frac{1}{2}\cos(2t)|_0^x = \frac{1}{2}\cos(2x) - \frac{1}{2}\cos(0).$$

So $a = \frac{1}{2}\cos(0) = \frac{1}{2}$.

3. Find the derivative of $\displaystyle\int_\pi^x \sec(t)dt$.

Solution: $\sec(x)$.

4. Find the derivative of $\displaystyle\int_\pi^x \sec(t)dt$.

Solution: $\sec(x)$.

5. Find the derivative of $\displaystyle\int_x^3 e^{-t^2} dt$.

Solution: $-e^{-x^2}$.

6. Find the derivative of $\int_0^{x^3} \ln(t)dt$.

Solution: $\ln(x^3) \cdot 3x^2$.

7. Find the derivative of $\int_{2x}^{x^2} \sqrt{t^2+t}dt$.

Solution: $\sqrt{(x^2)^2+x^2} \cdot 2x - \sqrt{(2x)^2+2x} \cdot 2 = 2x\sqrt{x^4+x^2} - 2\sqrt{4x^2+2x}$.

2 U-Substitution/Integration by Parts

8. True **FALSE** When integrating by parts, choosing different functions for u and dv (assuming both work out), will give different answers.

9. **TRUE** False It is always good to u sub first in order to simplify the integral.

Solution: This is true and u subbing first will make your life a lot easier.

10. Integrate $\int x(3x^2-5)^5dx$.

Solution: Let $u = 3x^2 - 5$ so $du = 6xdx$ so $xdx = \frac{du}{6}$. Replacing the things in the integral gives

$$\int x(3x^2-5)^5dx = \int u^5du/6 = \frac{u^6}{36} + C = \frac{(3x^2-5)^6}{36} + C.$$

11. Integrate $\int 2x^3e^{x^2}dx$.

Solution: Let $u = x^2$ so $du = 2xdx$. So

$$\int 2x^3e^{x^2}dx = \int ue^u du.$$

Now we use integration by parts. Let $r = u$ and $ds = e^u du$ so $s = e^u$ and $dr = du$. We have that

$$\int ue^u du = ue^u - \int e^u du = ue^u - e^u + C = x^2e^{x^2} - e^{x^2} + C.$$

12. Find $\int_0^1 \sqrt{1-\sqrt{x}}dx$.

Solution: We guess that $u = \sqrt{1-\sqrt{x}}$ and hence $u^2 = 1-\sqrt{x}$ so $\sqrt{x} = 1-u^2$ and $x = (1-u^2)^2 = u^4 - 2u^2 + 1$. Thus, we have that $dx = (4u^3 - 4u)du$. When $x = 0$, then $u = \sqrt{1-0} = 1$ and when $x = 0$, then $u = \sqrt{1-\sqrt{1}} = 0$. Thus

$$\int_0^1 \sqrt{1-\sqrt{x}}dx = \int_2^0 u(4u^3 - 4u)du = \frac{4}{5}u^5 - \frac{4}{3}u^3\Big|_1^0 = (0-0) - (4/5 \cdot 1^5 - 4/3 \cdot 1^3)$$
$$= \frac{4}{3} - \frac{4}{5}.$$

13. Find $\int x^5e^{x^3}dx$.

Solution: We guess that $u = x^3$ and so $du = 3x^2$. So $x^5e^{x^3} = x^3/3e^u du = u/3e^u du$ so

$$\int x^5e^{x^3}dx = \int \frac{ue^u}{3}du.$$

We use integration by parts to get that this is equal to

$$\frac{ue^u - e^u + C}{3} = \frac{x^3e^{x^3} - e^{x^3}}{3} + C.$$

14. Integrate $\displaystyle\int 2x^3\cos(x^2)dx$.

Solution: Let $u = \cos(x^2)$ and $dv = 2x^3dx$ so $du = -\sin(x^2) \cdot 2xdx$ and $v = \frac{x^4}{2}$. Then

$$\int 2x^3 \cos(x^2)dx = \frac{x^4 \cos(x^2)}{2} - \int -\sin(x^2)x^5dx.$$

This hasn't been simplified at all so we should try another approach. Instead, first u sub to get $u = x^2$ and $dx = 2x$ so $2x^3dx = 2xdx(x^2) = udu$. Therefore, we get

$$\int 2x^3 \cos(x^2)dx = \int u\cos(u)du = u\sin(u) + \cos(u) + C = x^2 \sin(x^2) + \cos(x^2) + C.$$

15. Integrate $\int 2x \arctan(x)dx$.

Solution: Let $u = \arctan(x)$ and $dv = 2xdx$ so $v = x^2$. Then

$$\int 2x\arctan(x)dx = x^2 \arctan(x) - \int \frac{x^2dx}{1+x^2} = x^2 \arctan(x) - \int \frac{x^2+1}{x^2+1} - \frac{1}{x^2+1}dx$$

$$= x^2 \arctan(x) - x + \arctan(x) + C.$$

16. Integrate $\int \frac{\ln \sqrt{x}}{\sqrt{x}}dx$.

Solution: Let $u = \ln(\sqrt{x})$ and $dv = \frac{dx}{\sqrt{x}} = x^{-1/2}dx$ so $v = 2x^{1/2} = 2\sqrt{x}$. Then by chain rule, we have $du = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}dx = \frac{1}{2x}dx$. Thus, we have

$$\int \frac{\ln\sqrt{x}}{\sqrt{x}}dx = 2\sqrt{x}\ln\sqrt{x} - \int 2\sqrt{x} \cdot \frac{dx}{2x} = 2\sqrt{x}\ln\sqrt{x} - \int \frac{dx}{\sqrt{x}} = 2\sqrt{x}\ln\sqrt{x} - 2\sqrt{x} + C.$$

17. Integrate $\int_0^{\pi/2} \sin(x)\cos(x)\sin(\sin(x))dx$.

Solution: First we u sub with $u = \sin(x)$ so $du = \cos(x)dx$ and this integral is equal to

$$\int_0^1 u\sin(u)du = -u\cos(u)\ |_0^1 + \int_0^1 \cos(u)du = -\cos(1) + \sin(1).$$

18. Integrate $\int_0^1 2x^3\sin(x^2)dx$.

Solution: First u sub with $u = x^2$ to get

$$\int_0^1 2x^3\sin(x^2)dx = \int_0^1 u\sin(u)du = -\cos(1)\sin(1).$$

The work is shown above.

19. Integrate $\int_0^1 x^{-1/2}\arctan(\sqrt{x})dx$.

Solution: First we u sub to make this problem easier. Let $u = \sqrt{x}$ so $du = \frac{1}{2\sqrt{x}}dx$ so $dx = 2\sqrt{x}du$. When $x = 0$, then $u = \sqrt{0} = 0$ and when $x = 1$, then $u = \sqrt{1} = 1$ so, we have that

$$\int_0^1 x^{-1/2}\arctan(\sqrt{x})dx = \int_0^1 2\arctan(u)du.$$

Now we can use integration by parts to get $r = \arctan(u)$ and $dt = 2du$ so $t = 2u$ and $dr = \frac{1}{1+u^2}$ so

$$\int_0^1 2\arctan(u)du = 2u\arctan(u)\ |_0^1 - \int_0^1 \frac{2u}{1+u^2}du.$$

Now we can v sub the second integral. Let $v = 1+u^2$ so $dv = 2udu$. So, we have that the integral is equal to

$$2\cdot 1\cdot\arctan(1) - 2\cdot 0\cdot\arctan(0) - \int_{u=0}^{u=1}\frac{1}{v}dv = 2\arctan(1) - \ln|1+u^2|_0^1 = 2\arctan(1) - \ln 2 = \frac{\pi}{2} - \ln 2.$$

20. Integrate $\int_1^{e^\pi}\sin(\ln(x))dx$.

Solution: First u sub to get that $u = \ln(x)$ so $du = \frac{1}{x}dx$ and $dx = xdu = e^u du$. The new bounds become $\ln 1$ and $\ln e^\pi$ which are 0 and π. Thus, we have that the integral is equal to

$$\int_0^\pi e^u\cos(u)du.$$

Now we can integrate by parts with $r = \sin(u)$ and $dt = e^u du$ so $dr = \cos(u)du$ and $t = e^u$ to get

$$\int_0^\pi e^u\sin(u)du = e^u\sin(u)|_0^\pi - \int_0^\pi \cos(u)e^u du = -\int_0^\pi \cos(u)e^u du.$$

We can integrate by parts again to get that the integral is equal to

$$\int_0^\pi e^u \sin(u)du = -\cos(u)e^u \mid_0^\pi - \int_0^\pi \sin(u)e^u du.$$

So the integral is equal to

$$\int_0^\pi e^u \sin(u)du = \frac{e^\pi + e^0}{2} = \frac{e^\pi + 1}{2}.$$

3 Symmetry

21. Is $f(x) = \dfrac{x\sin(x)}{x^2+4}$ even, odd, or neither?

Solution: Even.

22. Is $f(x) = x^2\tan(x)$ even, odd, or neither?

Solution: Odd.

23. Is $f(x) = xe^x$ even, odd, or neither?

Solution: Neither.

24. Is $f(x) = e^{x^2}\sin(x)$ even, odd, or neither?

Solution: Odd.

4 Numerical Integration

25. True **FALSE** Numerical approximations are just approximations, and never the exact answer.

Solution: Any approximation for a constant will give the exact answer.

26. **TRUE** False The second derivative can tell us if the midpoint rule gives an over/under estimate.

Solution: If the second derivative is always positive, then the midpoint rule gives an overestimate, and if the second derivative is always negative, the midpoint rule gives an underestimate.

27. **TRUE** False Simpson's method will approximate cubics exactly.

Solution: The error bound is given by K_4, which is the maximum of the fourth derivative. Since the fourth derivative of cubics is 0, the error is 0.

28. True **FALSE** When calculating K_1 of $f(x)$ on $[a, b]$, we have that K_1 is the maximum of $|f'(a)|$ and $|f'(b)|$.

Solution: K_1 is the maximum of $|f'(x)|$ on the interval $[a, b]$, which may not occur at the endpoints.

29. How many intervals do we need to use to approximate $\int_1^4 \ln x dx$ within $0.001 = 10^{-3}$ using Simpson's rule? Approximate it using Simpson's rule and $n = 4$.

Solution: We have that $E_S = \frac{K_4(b-a)^5}{180n^4}$ and $(\ln x)^{(4)} = \frac{-6}{x^4}$ so $10^{-3} = \frac{6(3)^5}{180n^4}$ so $n \geq 9.49$. So the minimum number is $n = 10$.

For $n = 4$, we have $\Delta x = \frac{4-1}{4} = \frac{3}{4}$ and our subintervals are $[1, 1.75], [1.75, 2.5], [2.5, 3.25], [3.25, 4]$. Simpson's rule gives us

$$\frac{3}{4 \cdot 6}(\ln 1 + 4 \ln 1.75 + 2 \ln 2.5 + 4 \ln 3.25 + \ln 4).$$

30. How many intervals do we need to use to approximate $\int_{-3}^{-1} 1/x^2 dx$ within $0.001 = 10^{-3}$ using the midpoint rule? Approximate it using the midpoint rule and $n = 4$.

Solution: We have that $E_M = \frac{K_2(b-a)^3}{24n^2}$ and $(1/x^2)^{(2)} = \frac{6}{x^4}$ so $10^{-3} \geq \frac{6(2)^3}{24n^2}$ so $n \geq 44.7$. So the minimum number is $n = 45$.

For $n = 4$, we have $\Delta x = \frac{-1-(-3)}{3} = \frac{2}{4}$ and our subintervals are $[-3, -2.5], [-2.5, -2], [-2, -1.5], [-1.5$ Midpoint rule gives us

$$\frac{1}{2}(1/(-2.75)^2 + 1/(-2.25)^2 + 1/(-1.75)^2 + 1/(-1.25)^2).$$

31. How many intervals do we need to use to approximate $\int_0^4 e^x dx$ within $0.001 = 10^{-3}$ using the trapezoid rule? Approximate it using the trapezoid rule and $n = 4$.

Solution: We have that $E_T = \frac{K_2(b-a)^3}{12n^2}$ and $(e^x)^{(2)} = e^x$ so $10^{-3} \geq \frac{e^4(4)^3}{12n^2}$ so $n \geq 17.06$. So the minimum number is $n = 18$.

For $n = 4$, we have $\Delta x = \frac{4-0}{4} = 1$ and our subintervals are $[0, 1], [1, 2], [2, 3], [3, 4]$. Trapezoid rule gives us

$$\frac{1}{2}(e^0 + 2e^1 + 2e^2 + 2e^3 + e^4).$$

32. Approximate the integral $\int_1^3 \frac{dx}{x}$ with $n = 2$ intervals using the different methods (left endpoint, right endpoint, midpoint, trapezoid, Simpson's).

Solution: Using the methods in order are

$$\frac{1}{1} \cdot 1 + \frac{1}{2} \cdot 1$$

$$\frac{1}{2} \cdot 1 + \frac{1}{3} \cdot 1$$

$$\frac{1}{1.5} \cdot 1 + \frac{1}{2.5} \cdot 1$$

$$\frac{1}{2}\left[\frac{1}{1} + 2 \cdot \frac{1}{2} + \frac{1}{3}\right]$$

$$\frac{1}{6}\left[\frac{1}{1} + 4 \cdot \frac{1}{2} + \frac{1}{3}\right]$$

33. What is the smallest value of n needed to ensure that our numerical approximation method for $\int_1^3 dx/x$ is within $0.0001 = 10^{-4}$ using the different methods?

Solution: In order to do this, you need to set the error bound less than equal to this bound.

$$E_L = \frac{K_1(b-a)^2}{2n} = \frac{1(3-1)^2}{2n} \leq 10^{-4} \implies n \geq 2 \cdot 10^4.$$

$$E_R = \frac{K_1(b-a)^2}{2n} = \frac{1(3-1)^2}{2n} \leq 10^{-4} \implies n \geq 2 \cdot 10^4.$$

$$E_M = \frac{K_2(b-a)^3}{24n^2} = \frac{2(3-1)^2}{24n^2} \leq 10^{-4} \implies n \geq \frac{100}{\sqrt{3}}.$$

$$E_T = \frac{K_2(b-a)^3}{12n^2} = \frac{2(3-1)^2}{12n^2} \leq 10^{-4} \implies n \geq \frac{100\sqrt{2}}{\sqrt{3}}.$$

$$E_S = \frac{K_4(b-a)^5}{180n^4} = \frac{24(3-1)^5}{180n^4} \leq 10^{-4} \implies n \geq \frac{20\sqrt{2}}{\sqrt[4]{15}}.$$

5 Improper Integrals

34. True **FALSE** We can compare an integral to $\int_1^\infty 1/\sqrt{x}dx$ in order to show it converges.

Solution: The given integral diverges and hence cannot be used to show an integral converges.

35. True **FALSE** We can compare an integral to $\int_1^\infty 1/x^2 dx$ to show it diverges.

Solution: The given integral converges and hence cannot be used to show that another integral diverges.

36. True **FALSE** Since $x < x+1$, we have that $\infty = \int_1^\infty \frac{1}{x}dx \leq \int_1^\infty \frac{1}{x+1}dx$ so the latter integral diverges.

Solution: When we take reciprocals, we need to switch the sign so we actually get $\infty \int_1^\infty \frac{1}{x}dx \geq \int_1^\infty \frac{1}{x+1}dx$ so we don't have any information on if the latter integral converges or not. It does in fact diverge but we need to show that a different way.

37. Calculate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+(x-1)^3}dx.$

Solution: We have to split up the integral first but it doesn't matter where we do so. We choose $x = 1$ for simplicity.

$$\int_{-\infty}^{\infty} \frac{1}{1+(x-1)^2}dx = \int_{-\infty}^{1} \frac{1}{1+(x-1)^2}dx + \int_{1}^{\infty} \frac{1}{1+(x-1)^2}dx$$

$$= \lim_{t\to-\infty} \int_t^1 \frac{1}{1+(x-1)^2}dx + \lim_{r\to\infty} \int_1^r \frac{1}{1+(x-1)^2}dx = \lim_{t\to-\infty} \arctan(x-1)|_t^1 + \lim_{r\to\infty} \arctan(x-1)|_1^r$$

$$= \arctan(0) - \arctan(-\infty) + \arctan(\infty) - \arctan(0) = \pi/2 - (-\pi/2) = \pi.$$

38. Calculate $\displaystyle\int_1^\infty xe^{-2x}dx.$

Solution: First we calculate that

$$\int xe^{-2x}dx = x\frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}dx = \frac{-xe^{-2x}}{2} - \frac{e^{-2x}}{4} + C,$$

by integration by parts. Thus, we have that

$$\int_1^\infty xe^{-2x}dx = \lim_{t\to\infty} \int_1^t xe^{-2x}dx = \lim_{t\to\infty} \left[\frac{-xe^{-2x}}{2} - \frac{e^{-2x}}{4} \right|_1^t$$

$$= \lim_{t\to\infty} \left[\frac{-te^{-2t}}{2} - \frac{e^{-2t}}{4} + \frac{e^{-2}}{2} + \frac{e^{-2}}{4} \right] = \frac{e^{-2}}{2} + \frac{e^{-2}}{4}.$$

This is because we can use L'Hopital's rule to calculate that $\lim_{t\to\infty} te^{-t} = 0$.

39. Calculate $\displaystyle\int_1^\infty \frac{2x}{1+x^2}dx.$

Solution: We have that

$$\int_1^\infty \frac{2x}{1+x^2}dx = \lim_{t\to\infty} \int_1^t \frac{2x}{1+x^2}dx = \lim_{t\to\infty} \ln(1+x^2)|_1^t = \infty.$$

40. Does $\int_3^\infty \frac{1}{\sqrt{x}\ln(x)}$ converge?

Solution: We know that for $x \geq 3$ that $x \geq \sqrt{x}$ and so $x\ln(x) \geq \sqrt{x}\ln(x)$ so $\frac{1}{x\ln(x)} \leq \frac{1}{\sqrt{x}\ln(x)}$ so

$$\int_3^\infty \frac{1}{\sqrt{x}\ln(x)} \geq \int_3^\infty \frac{1}{x\ln(x)}dx = \infty.$$

So this integral diverges.

41. Does $\int_1^\infty \frac{2x+2xe^{-x}}{1+x^2}dx$ converge?

Solution: We know that $1+e^{-x} \geq 1$ and so $2x(1+e^{-x}) \geq 2x$ and so $\frac{2x+2xe^{-x}}{1+x^2} \geq \frac{2x}{1+x^2}$ and

$$\int_1^\infty \frac{2x+2xe^{-x}}{1+x^2}dx \geq \int_1^\infty \frac{2x}{1+x^2}dx = \infty.$$

So, the integral diverges.

6 Partial Fractions

42. Integrate $\int \frac{5x+17}{x^2+2x-15}dx$.

Solution: We have that $\frac{5x+17}{x^2+2x-15} = \frac{5x+17}{(x-3)(x+5)} = \frac{A}{x-3} + \frac{B}{x+5}$. Multiplying gives $5x + 17 = A(x+5) + B(x-3)$ and plugging in $x = 3$ and $x = -5$ gives $32 = 8A$ and $-8 = -8B$ respectively or $A = 4, B = 1$ and hence

$$\int \frac{5x+17}{x^2+2x-15}dx = \int \frac{4}{x-3} + \frac{1}{x+5}dx = 4\ln|x-3| + \ln|x+5| + C.$$

43. Integrate $\int \frac{2x^3-12x^2+28x-23}{(x-2)^2(x-1)^2}dx$.

Solution: We set $\frac{2x^3-12x^2+28x-23}{(x-2)^2(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2} + \frac{D}{(x-2)^2}$. Multiplying through and solving gives us $A = 0, B = -5, C = 2, D = 1$. Thus, we have

$$\int \frac{2x^3-12x^2+28x-23}{(x-2)^2(x-1)^2}dx = \int \frac{2}{x-2} + \frac{1}{(x-2)^2} - \frac{5}{(x-1)^2}dx$$

$$= 2\ln|x-2| - \frac{1}{x-2} + \frac{5}{x-1} + C.$$

44. Set up the partial fraction decomposition of $\frac{3x^2+1}{(x-1)(x^2+4)^2(x^2+2x+2)^2}$ (you don't have to solve for the coefficients).

Solution: Since $x^2 + 2x + 2$ is irreducible, we have

$$\frac{3x^2 + 1}{(x-1)(x^2+4)^2(x^2+2x+2)^2} = \frac{A}{x-1} + + \frac{Bx + C}{x^2 + 4} + \frac{Dx + E}{(x^2+4)^2} + \frac{Fx + G}{x^2 + 2x + 2} + \frac{Hx + J}{(x^2 + 2x + 2)^2}.$$